**Sequel to “Partitioning Transaction Vectors into Pure Investments”:**

**A New Sufficient Condition for Transactions to have a Unique IRR; and Some Results on the Distribution of IRRs.**

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*This is the accepted author version of the article published online in The Engineering Economist, DOI: 10.1080/0013791X.2021.1929624, on 20 July 2021.*

Abstract.

An article by the same author in The Engineering Economist in 2018, (vol 63(2), 143-157), proved that any transaction could be uniquely partitioned into a sequence of pure investments with strictly decreasing IRRs. This article uses that result to prove a new condition for a transaction to have a unique IRR: and also to give some information on how the IRRs of a transaction must be distributed.

Key words: investment theory; internal rate return

**1. Introduction.**

It was proved by Cuthbert in an article published in the Engineering Economist in 2018, (Cuthbert, 2018), that any transaction vector could be uniquely partitioned into a sequence of pure investments with strictly decreasing internal rates of return, (IRR). The introduction to that article put that result into the context of earlier research: and that context material will not be repeated here.

What this article does is to establish some consequences of Cuthbert’s partitioning theorem. First of all, in the form of a new sufficient condition for a transaction to have a unique IRR. And secondly by establishing, for transactions where the IRR is not unique, some results on how IRRs are distributed.

**2. Notation, and the fundamental partitioning theorem proved in Cuthbert (2018).**

The notation is as in Cuthbert, (2018), and is as follows.

A transaction **a** is defined as a vector $\left(a\_{0}, a\_{1}, …. , a\_{n}\right)$ , where negative terms represent investments of capital, and positive terms represent repayments. Without loss of generality, it is assumed that $a\_{0}<0$ .

The net present value, (NPV), of the payment stream **a**, calculated at discount rate u, (u>-1), and with year 0 reference date, is defined, as in the standard definition, as

 

An IRR  of **a,** where **a** is a payment stream,is defined as any discount rate  such that

  .

Where **a** is a transaction vector, and is an IRR of a, then the invested capital, , at the start of period k is defined as the negative of the accumulated value at time k-1 of the payments up to time k-1: that is,

 , and  , k = 1,… n.

This definition implies the following recursive relationship:

 , and , (j 1).

It can readily be shown that .

An important special class of transactions, known as pure investments, are those where the invested capitals, $d\_{j}$, are all non-negative. More formally, if **a** is a transaction vector, with IRR , and if  for all j, **a** is defined to be a pure investment. It turns out, in a result proved originally by Soper, (1959), and generalised by Gronchi, (1986), that every pure investment has exactly one IRR.

Cuthbert, (2018), gave a formal definition of the operation of partitioning a transaction: informally, this can be regarded as the process of splitting the transaction into non-overlapping segments.

More formally, if **a** is a vector then, for any k, the vectors (), () represent a partitioning of **a**. So, for example, (3,5,4) can be partitioned as (3,5) and (4).

This is denoted as the short form notation for a partition. Sometimes it will be convenient to retain the position of the partitioning vectors in the original n+1 length vector: so I will work with (3,5,0) and (0,0,4): I will call this the long form notation. One advantage of the long form notation is that it is additive: (3,5,4) = (3,5,0) + (0,0,4).

Finally, the standard definition of IRR, as given above, only applies for discount rates greater than -1. The convention was adopted, and justified, in Cuthbert, (2018), that if a vector consists only of terms which are negative or zero, (with at least one term strictly negative), then this corresponds to a pure investment transaction with IRR = -1.

Having dealt with these preliminaries, it is now possible to state the basic result proved in Cuthbert, (2018) as follows.

Partitioning Theorem.

Let **a** be a transaction vector. Then there exists a unique integer K $\geq $ 1, and a partition **a**(1), **a**(2), …., **a**(K) of **a**, where each of the **a**(1), …., **a**(K) are pure investments, and where

IRR(**a**(1)) > IRR(**a**(2)) > ….> IRR(**a**(K).

Moreover, the integer K, and the partition **a**(1), …., **a**(K), are unique.

**3. Preliminaries.**

The following sections will set out and prove the main results of this article. Before that, however, it is useful to introduce some further notation and preliminary theory, which will be required in the proofs of the main results.

Extremal Transactions.

In Cuthbert, (2018), a very simple transaction was defined as a transaction where there were only two non-zero terms, with a negative immediately preceding a positive term.

An extremal transaction of length (n+1) is defined here as a transaction which has a very simple transaction in positions 0 and 1, a very simple transaction in positions n-1 and n: and has all other terms equal to zero.

The derivative of the net present value function at 0.

If **a** is a transaction with IRR $σ$ , and invested capitals $d\_{j}$ , then it is a standard result, first proved by Hazen, (2003), that

 NPV(**a**, u) = ($σ$ – u) NPV(**d**, u) :

 (a proof of this was given in Cuthbert, (2018)).

Differentiating this expression, it follows that.

 $\acute{NPV\left(a, u\right)}$ = - NPV(**d**, u) + ($σ$ – u)$ \acute{NPV\left(d, u\right)}$

 = - $\sum\_{j=1}^{n}d\_{j}(1+u)^{-j}$ - $(σ-u)\sum\_{j=1}^{n}jd\_{j}(1+u)^{-j-1}$

Putting u = 0, it follows that

 $\acute{NPV\left(a, 0\right)}$ = - $\sum\_{j=1}^{n}d\_{j}$ - $σ\sum\_{j=1}^{n}jd\_{j}$

 = - $\sum\_{j=1}^{n}(1 + jσ)d\_{j}$ (1)

Note also from the above expression for NPV(**a**, u) that

 NPV(**a**, 0) = $\sum\_{j=0}^{n}a\_{j}$ = $σ\sum\_{j=1}^{n}d\_{j}$ (2)

The operation of scaling a transaction.

It will be useful in what follows to use the concept of scaling a transaction. This is defined here as follows. Let **a** be a transaction. Then, for any number $ρ>-1$, the transaction **a** scaled by $ρ$ is the new transaction, $a^{S}$ defined by

 $a\_{j}^{S}$ = $a\_{j}(1+ ρ)^{-j}$ .

The following properties of this scaling operation will be used in the proof.

(i) If $σ$ is an IRR of **a**, then $\frac{\left(1+σ\right)}{\left(1+ρ\right)}-1$ is an IRR of $a^{S} .$

(ii) The unique partition of $a^{S}$ into pure investments is the same as the partition of **a** .

(iii) If **a** has a unique partition into pure investments with IRRs $σ\_{1}$ , … $σ\_{K}$ , where $σ\_{K}> -1$: and if the quantity $τ$ = $τ(a)$ is defined as

 $τ$ = $\frac{(1+ σ\_{1})}{(1+ σ\_{K})}$ , then $τ$ is invariant under scaling of **a**.

 (iv) The derivative of the function NPV($a^{S}$, u), at u, has the same sign, (positive, negative, or zero), as the derivative of NPV(**a**, x) at x = (1+$ρ$)(1+u) – 1.

Proof. Proofs of the above properties are given in Appendix 1.

**4. Theorems on the uniqueness and distribution of IRRs.**

This section deals with transactions **a** where the final term in the transaction is positive: that is, where the IRR, $σ\_{K}$ , of the final term in the unique partition of **a** satisfies $σ\_{K}$ > -1. The next section deals with transactions where the final term in the transaction is negative.

In what follows, the quantity $τ$ = $τ(a)$ associated with a transaction is defined, as in the preceding section, as $τ$ = $\frac{(1+ σ\_{1})}{(1+ σ\_{K})}$ , where $σ\_{1}$ and $σ\_{K}$ are respectively the IRRs of the first and last terms in the unique partition of **a** into pure investments.

Theorem 1 gives a sufficient condition for **a** to have a unique IRR.

**Theorem 1.**

Let **a** be a transaction of length (n+1), with $a\_{0}<0$ and $a\_{n}$ > 0, and n $\geq $ 3.

Then if $τ \leq \frac{n^{2}}{(n-2)^{2}}$ , **a** has a unique IRR.

Proof.

The proof is conveniently considered jointly with that of Theorem 2.

Note that, since the case $τ $ = 1 corresponds to **a** itself being a pure investment, Theorem 1 can actually be regarded as a generalisation of Soper’s original uniqueness result, referred to above.

It was established in Cuthbert, (2018), that all of the IRRs of **a** must lie strictly between$ σ\_{K}$ and $σ\_{1}$ . When the sufficient condition of Theorem 1 is not met, Theorem 2 gives information on the distribution of IRRs in this interval.

**Theorem 2.**

Let **a** be a transaction of length (n+1), with $a\_{0}<0$ , and $a\_{n}>0$ , and $n \geq 3$ .

If $τ> \frac{n^{2}}{(n-2)^{2}}$ , then

(i) **a** has at most one IRR, $ρ$ say, such that

 (1 + $ρ$) $\geq $ $\frac{2(n-1)(1+ σ\_{1})}{(n+ τ\left(n-2)\right) - [(n + τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}$

(ii) **a** has at most one IRR, $ρ$ say, such that

 (1 + $ρ$) $\leq $ $\frac{2(n-1)(1+ σ\_{1})}{(n+ τ\left(n-2)\right)+ [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}$

(iii) if **a** has multiple IRRs, it has at least one IRR, $ρ$ say, such that

$\frac{2(n-1)(1+ σ\_{1})}{(n+ τ\left(n-2)\right)+ [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}} \leq \left(1+ ρ\right)\leq $ $\frac{2(n-1)(1+ σ\_{1})}{(n+ τ\left(n-2)\right) - [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}$ .

Proof.

The proofs of Theorems 1 and 2 are given together in Appendix 2.

The section concludes with some notes and observations on these results.

a) First of all, it is useful to elucidate the results graphically. The following chart gives a graphical illustration of the implications of Theorems 1 and 2.



This illustrates the specific case where $\left(1+ σ\_{1}\right)$ = 1.1 and $τ$ = 1.5, so that $\left(1+ σ\_{K}\right)$ = 0.7333.

The horizontal axis in the chart represents n. The two horizontal lines correspond to $\left(1+ σ\_{1}\right)$ and $\left(1+ σ\_{K}\right)$ , (and, of course, any IRR of **a** must be such that (1 + that IRR) lies between the two horizontal lines). The two curved lines represent the theorem upper and lower bounds.

Now, for n = 10, $\frac{n^{2}}{(n-2)^{2}}$ = 1.5625, which is greater than $τ$ :

And, for n = 11, $\frac{n^{2}}{(n-2)^{2}}$ = 1.494, which is less than $τ$.

So, from the sufficient condition in the theorem, any transaction with these values of $\left(1+ σ\_{1}\right)$ and $\left(1+ σ\_{K}\right)$ , and which has n less than or equal to 10, must have a unique IRR. This corresponds to the portion of the chart left of the curved lines.

For a transaction with these values of $\left(1+ σ\_{1}\right)$ and $\left(1+ σ\_{K}\right)$ , and which has n greater than or equal to 11, the transaction can have at most one IRR such that (1 + that IRR) is on or above the upper curved line: at most one IRR such that (1 + that IRR) is on or below the lower curved line: and if the transaction has more than one IRR, there must be at least one IRR such that (1 + that IRR) lies on or between the two curved lines.

b) The bounds in Theorem 2 can also be expressed in terms of $σ\_{K}$ , where $σ\_{K}$ is the smallest IRR in the unique partition of **a**. Using the fact that

$$\{(n+ τ\left(n-2)\right)+ [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}\}\{(n+ τ\left(n-2)\right) - \left[(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}\right]^{0.5}\}$$

 $=4τ(n-1)^{2}$ ,

It follows readily that the bounds in, for example, part (iii) of Theorem 2 can be expressed as $\frac{\{(n+ τ\left(n-2)\right) – \left[(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}\right]^{0.5}\}(1+ σ\_{K})}{2\left(n-1\right)} \leq \left(1+ ρ\right)$

 $\leq $ $\frac{\{(n+ τ\left(n-2)\right)+ \left[(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}\right]^{0.5}\}(1+ σ\_{K})}{2\left(n-1\right)} $.

c) The bounds in Theorem 2 cannot be tightened any further, as can be seen from the fact that the extremal transactions discussed in sub-section 7 of the proof lie on the bound.

d) As $τ $ tends to infinity, the upper bound in Part (iii) of Theorem 2 tends to

 $(1- \frac{1}{\left(n-1\right)})(1+ σ\_{1})$ , and the lower bound tends to zero.

Proof. The proof is given in Appendix 3

e) The bounds in Theorem 2 depend on the roots of a particular quadratic equation; namely, the quadratic at equation (12) in the proof of the Theorem. It follows from the argument in Appendix 3 that the lower root of this quadratic is always greater than $\frac{(n-1)}{(n-2)}$ . This means that a weaker version of Theorem 2 can be stated. For example, the weaker version of Theorem 2 part (iii) is:-

“Let **a** be a transaction of length (n+1), with $a\_{0}<0$ , and $a\_{n}>0$ , and $n \geq 3$ .

if **a** has multiple IRRs, it has at least one IRR, $ρ$ say, such that

 $(1+ \frac{1}{\left(n-2\right)})(1+ σ\_{K}) < \left(1+ ρ\right) <$ $(1- \frac{1}{\left(n-1\right)})(1+ σ\_{1})$ .”

 There are corresponding weaker versions of parts (i) and (ii).

The only reason for quoting this weaker version of the theorem is because it is so computationally simple it may on occasion be useful.

**5. The case where the final term in the transaction is negative.**

This section considers transactions for which $a\_{n}$ < 0. For such transactions the function NPV(**a**, u) is negative for small u, and for large u. It is possible for this type of transaction that **a** may not have any IRRs: this corresponds to the possibility that NPV(**a**, u) < 0 for all u. But if **a** does have one or more IRRs, it will always be the case that it has at least one IRR, $ρ$ say, for which the derivative of the NPV function at $ρ$ is non-negative.

In the case that **a** has at least one IRR, the following theorem holds.

**Theorem 3.**

Let **a** be a transaction of length (n+1), with $a\_{0}<0$ , and $a\_{n}<0$ , and $n \geq 2$ , and suppose that **a** has at least one IRR. Then

(i) **a** has at most one IRR, $ρ$ say, such that

 (1 + $ρ$) > $(1- \frac{1}{n})(1+ σ\_{1})$

(ii) **a** has at least one IRR, $ρ$ say, such that

 (1 + $ρ$) $\leq $ $(1- \frac{1}{n})(1+ σ\_{1})$

Proof.

The proof is given in Appendix 4.

It is worth restating one specific implication arising from Theorem 3. Namely, if a transaction with $a\_{n}$ < 0 has a unique IRR, that IRR must be less than or equal to the bound implicit in Theorem 3.

**6. Conclusion.**

In conclusion, the following two remarks appear relevant.

a) This article has shown how the partition theorem proved in Cuthbert, (2018), can be used to prove quite powerful results, both in the form of a new sufficient condition for a transaction to have a unique IRR: and also giving some information on how the IRRs must be distributed for transactions with multiple IRRs. It is interesting that the results proved in this article depend only on the IRRs of the first and last terms in the unique partition of the original transaction **a**: the results do not depend in any way on the relative sizes of the amounts of capital invested at different stages of the project’s life. It is an open question whether useful stronger results could be proved by bringing the amounts of capital invested into consideration.

b) The results proved here perhaps throw some oblique light on the argument put forward by Ben-Horin and Kroll in their 2012 article in the Engineering Economist : (Ben-Horin and Kroll, 2012). That article was titled “The limited relevance of the multiple IRRs” – and argued that, in terms of the types of transaction likely to be encountered in practice, problems of multiple IRRs were likely to be relatively uncommon. It is interesting, however, that the specific case solved by Ben-Horin and Kroll in their Proposition 2 deals with transactions of length 4, (i.e., n = 3 in the notation of this article): and the same is true of the numerical example they use to illustrate that proposition. But as can be seen from Theorem 1 in the present article, in the n = 3 case a transaction must have a $τ$ value greater than $\frac{3^{2}}{(3-2)^{2}}$ = 9, for it to have any possibility of having multiple IRRs: (and indeed, Ben-Horin and Kroll’s numerical example of a transaction with multiple IRRs has a $τ$ value which can be computed as 9.05). Taking a transaction of length 4, therefore, necessarily implies wide variability in the individual cash flows before any transaction has a chance of having multiple IRRs. But given that the quantity $ \frac{n^{2}}{(n-2)^{2}}$ in Theorem 1 decreases quite rapidly with increasing n, it is not obvious that wide variability in cash flows will always be required for a transaction with large n to have multiple IRRs. Perhaps Ben-Horin and Kroll’s choice of n=3 to illustrate their argument was, unwittingly, not the most appropriate choice to have made.

**Acknowledgement.**

I am very grateful to Carlo Alberto Magni for helpful comments during the research which led to the preparation of this paper.

**Funding Statement.**

This research did not receive any specific grant from funding agencies in the public, commercial or not-for-profit sectors.

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**Appendices.**

**Appendix 1: proof of properties of scaling in Section 3.**

(i) If $σ$ is an IRR of **a**, then $\frac{\left(1+σ\right)}{\left(1+ρ\right)}-1$ is an IRR of $a^{S}$

Proof. NPV($a^{S}, \frac{\left(1+σ\right)}{\left(1+ρ\right)}-1)$ = $\sum\_{j=1}^{n}a\_{j}(1+ ρ)^{-j}(1+ ρ)^{j}(1+ σ)^{-j}$ = $\sum\_{j=1}^{n}a\_{j}(1+ σ)^{-j}$ = 0 .

(ii) The unique partition of $a^{S}$ into pure investments is the same as the partition of **a** .

Proof. The first step is to prove that, if **a** is a pure investment then so is $a^{S}$ .

Suppose that **a** has IRR $σ$ . then, by (i), $a^{S}$ has an IRR equal to $\frac{\left(1+σ\right)}{\left(1+ρ\right)}-1$.

Using the formula $d\_{j+1}= -\sum\_{k=0}^{j}(1+σ)^{j-k}a\_{k}$ , and applying this to $a^{S}$ , it follows that

 $d\_{j+1}^{S} $= $-\sum\_{k=0}^{j}\frac{(1+σ)^{j-k}}{(1+ρ)^{j-k}}a\_{k}^{S}$

 = $-\sum\_{k=0}^{j}\frac{(1+σ)^{j-k}}{(1+ρ)^{j-k}}(1+ρ)^{-k}a\_{k}$

 =$ -(1+ρ)^{-j}\sum\_{k=0}^{j}(1+σ)^{j-k}a\_{k}$

 = $(1+ρ)^{-j}d\_{j+1}$ .

Since the terms $(1+ρ)^{-j}$ are positive, it follows that $d\_{j} \geq 0$ implies the same is true for the invested capital of the scaled investment: hence if **a** is a pure investment, so is $a^{S}$ .

It follows that if **a** has a unique partition into pure investments with IRRs $σ\_{1}$ , … $σ\_{K}$ , then the terms in the same partition of $a^{S}$ are also pure investments, with IRRs

 $\frac{(1+σ\_{1})}{(1+ ρ)}-1$ , … $\frac{(1+σ\_{K})}{(1+ ρ)}-1$ . Since this is a decreasing sequence, this partition of $a^{S}$ is therefore the unique partition.

(iii) If **a** has a unique partition into pure investments with IRRs $σ\_{1}$ , … $σ\_{K}$ , where $σ\_{K}> -1$: and if the quantity $τ$ = $τ(a)$ is defined as

 $τ$ = $\frac{(1+ σ\_{1})}{(1+ σ\_{K})}$ , then $τ$ is invariant under scaling of **a**.

Proof. $τ(a^{S})$ = $\frac{(1+σ\_{1})}{(1+ρ)}\frac{(1+ρ)}{(1+σ\_{K})}$ = $\frac{(1+σ\_{1})}{(1+σ\_{K})}$ = $τ(a)$ .

(iv) The derivative of the function NPV($a^{S}$ , u), at u, has the same sign, (positive, negative, or zero), as the derivative of NPV(**a**, x) at x = (1+$ρ$)(1+u) – 1.

Proof. NPV$(a^{S}, u)$ = $\sum\_{j=0}^{n}a\_{j}(1+ρ)^{-j}(1+u)^{-j}$ = NPV(**a**, (1+$ρ$)(1+u) – 1).

Differentiating both sides with respect to u, it follows that

 $\acute{NPV} (a^{S}, u)$ = (1+$ρ$) $\acute{NPV}$ (**a**, (1+$ρ$)(1+u) – 1).

Since (1+$ρ$) > 0, the result follows.

**Appendix 2: Proof of Theorems 1 and 2 in Section 4**

1. Suppose for the remainder of this proof that the transaction **a** has multiple IRRs. Since the function NPV(**a**, u) is a continuous and differentiable function for u > -1, it follows from an elementary continuity argument that **a** cannot have two consecutive IRRs at which the derivative of the NPV function is strictly negative at both. (Because, suppose not. Let the two IRRs be x and y, with x < y. Then since the NPV function is downward sloping at both x and y, the NPV function must be less than zero in some neighbourhood immediately greater than x, and greater than zero in some neighbourhood immediately less than y. By continuity, this implies the NPV must equal zero at some point between x and y: which contradicts the assumption that x and y are consecutive IRRs.)

It follows, since **a** is assumed to have more than one IRR, that it has at least one IRR, $ρ$ say, for which $\acute{NPV\left(a, ρ\right)} \geq 0 $.

2. Let $ρ$ be some IRR of **a** for which $\acute{NPV\left(a, ρ\right)} \geq 0 $: (such a $ρ$ exists, by (1)). Now, recalling the concept of scaling introduced in Section 3, scale **a** by $ρ$. Then, by the properties of scaling as set out in section 3 scaled **a** has an IRR at $\frac{(1+ρ)}{(1+ρ)}-1$ = 0, and the derivative of the NPV function of scaled **a** at 0 is $\geq $ 0. Further, scaled **a** has the same partition as **a** into pure investments: and the $τ$ value of scaled **a** is the same as the $τ$ value of **a**.

3. Without loss of generality, therefore, it will be assumed that we are dealing with a transaction **a** which has already been subject to such a scaling. (This assumption will be relaxed at the penultimate stage of the proof.) In other words, we assume we are dealing with a transaction **a** for which 0 is an IRR of **a**, and $\acute{NPV\left(a, 0\right)} \geq 0$.

Let **a**(1), **a**(2), …., **a**(K) be the unique partition of **a** into pure investments, and let $π\_{1}$, … , $π\_{K}$ be the corresponding sequence of strictly decreasing IRRs. (Note that, since 0 is an IRR of **a**, $π\_{1}>0$, and $π\_{K}<0$ .)

Let **d**(j) be the vector of invested capital of **a**(j), (calculated at the IRR $π\_{j}$ of **a**(j)): and let **d** be the (n+1) vector (**d**(1), **d**(2),… , **d**(K)).

Further, define the (n+1) vector $σ$as follows:-

 $σ\_{j}$ = $π\_{i}$ , where i is the element of the partition of **a** to which j belongs:

So $σ$= ( $π\_{1}$ **,** ,….. , $π\_{1}$, $π\_{2}$,… , $π\_{2}$,………, $π\_{K}$,…. , $π\_{K}$) , partitioned in the same way as **a**.

If each **a**(j) is written in long form notation, so that **a** = $\sum\_{j=1}^{K}a(j)$, and if the result in formula (1) in Section 3 is applied to each of the terms in the summation, it follows that $\acute{NPV\left(a, 0\right)} $is given by the expression

 $\acute{NPV\left(a, 0\right)} $ = $- \sum\_{j=1}^{n}(1+jσ\_{j})d\_{j}$ (3)

(4) What will now be proved is one of the two key steps in this proof: the other key step being in the following sub-section. What will be shown in this sub-section is that it is possible to find an extremal transaction, $\hat{a}$ , of length (n+1), and with the following properties: namely,

 0 is an IRR of $\hat{a}$**:**

 $τ\left(\hat{a}\right)= τ(a)$

 $\acute{NPV\left(\hat{a}, 0\right)} \geq $ $\acute{NPV\left(a, 0\right)}$,

with the latter inequality being strict, other than in the trivial case where **a** itself is extremal.

This is proved by construction, as follows:

The vector $σ$ , defined in sub-section (3) above, has terms $σ\_{j}$ which are non-increasing, with $σ\_{0}= π\_{1} >0$, and $σ\_{n}= π\_{K }< 0$ . It follows that there exists a uniquely defined k such that

 $σ\_{j }\geq 0 $ for all j < k, and

 $σ\_{j }< 0$ for all j $\geq $ k.

Then define the vector $\hat{a}$ as follows:-

 $\hat{a\_{0}}$ = $-\frac{1}{σ\_{0}}\sum\_{j=0}^{k-1}a\_{j}$

 $\hat{a\_{1}}$ = $\frac{(1+σ\_{0})}{σ\_{0}}\sum\_{j=0}^{k-1}a\_{j}$

 $\hat{a\_{n-1}}$ = $-\frac{1}{σ\_{n}}\sum\_{j=k}^{n}a\_{j}$

 $\hat{a\_{n}}$ = $\frac{(1+σ\_{n})}{σ\_{n}}\sum\_{j=k}^{n}a\_{j}$

 $\hat{a\_{j}}$ = 0 for all other j.

Applying formula (2) above to each of the pure investments in the partition of **a**, it follows that

 $\hat{a\_{0}}$ = $-\frac{1}{σ\_{0}}\sum\_{j=1}^{k-1}σ\_{j}d\_{j}$ < 0:

and $\hat{a\_{n-1}}$ = $-\frac{1}{σ\_{n}}\sum\_{j=k}^{n}σ\_{j}d\_{j}$ < 0.

So $\hat{a}$is a properly defined extremal transaction, and since $σ\_{0}$ = $π\_{1}$, and $σ\_{n}$ = $π\_{K}$ , it has the same $τ$ value as the original **a**.

Further, NPV($\hat{a}$**,** 0) **=** $\frac{σ\_{0}}{σ\_{0}}\sum\_{j=0}^{k-1}a\_{j}$ + $\frac{σ\_{n}}{σ\_{n}}\sum\_{j=k}^{n}a\_{j}$ = $\sum\_{j=0}^{n}a\_{j}$ = 0, since 0 is an IRR of **a**: so 0 is also an IRR of $\hat{a}$ **.**

Now, since NPV($\hat{a}$**,** u)can be written as

 NPV($\hat{a}$**,** u) = $\hat{a\_{0}}$ – (1+$σ\_{0}) \hat{a\_{0}}(1+u)^{-1}$ + $\hat{a\_{n-1}}(1+u)^{-(n-1)}$ – (1+$σ\_{n}) \hat{a\_{n-1}}(1+u)^{-n}$,

it follows readily that

 $\acute{NPV\left(\hat{a}, 0\right)} $ = $(1+σ\_{0}) \hat{a\_{0}}$ + (1 + n$σ\_{n}$)$ \hat{a\_{n-1}}$

 = - $\frac{(1+σ\_{0})}{σ\_{0}}\sum\_{j=1}^{k-1}σ\_{j}d\_{j}$ - $\frac{(1+nσ\_{n})}{σ\_{n}}\sum\_{j=k}^{n}σ\_{j}d\_{j}$ .

Consider the coefficient of $d\_{j}$ in the first summation in this expression. There are two cases to consider.

(a) If $σ\_{j}$ > 0, then this coefficient

 = $- \frac{(1+σ\_{0})}{σ\_{0}}σ\_{j}$

 = $- (1+\frac{1}{σ\_{0}})σ\_{j}$

 $\geq $ $- (1+\frac{1}{σ\_{j}})σ\_{j}$ , since $σ\_{j} \leq σ\_{0}$ ,

 = $- (1+ σ\_{j})$

 $\geq $ $- (1+ jσ\_{j})$ , with strict inequality if j > 1.

(b) If $σ\_{j}$ = 0, then the coefficient

 = $- \frac{(1+σ\_{0})}{σ\_{0}}σ\_{j}$ = 0 > -1 = $- (1+ jσ\_{j})$ .

Now consider the coefficient of $d\_{j}$ in the second summation in the expression for $\acute{NPV\left(\hat{a}, 0\right)} $. This coefficient

 = $- \frac{(1+nσ\_{n})}{σ\_{n}}σ\_{j}$

 = $- (\frac{1}{σ\_{n}} + n)σ\_{j}$

 $\geq - (\frac{1}{σ\_{j}} + n)σ\_{j}$ , since - $σ\_{j}$ > 0

 = $- (1 + nσ\_{j})$

 $\geq $ - (1 +j$σ\_{j}$) , (again, since - $σ\_{j}$ > 0),

 and with strict inequality if j < n.

Hence $\acute{NPV\left(\hat{a}, 0\right)} $ $\geq $ $- \sum\_{j=0}^{n}(1+jσ\_{j})d\_{j}$ = $\acute{NPV\left(a, 0\right)}$ , (by equation (3) above), with strict inequality except in the trivial case where **a** itself is extremal. This completes this subsection of the proof.

5) This sub-section is the second key step in the proof. What the sub-section does is to explore the conditions which the terms of the extremal transaction $\hat{a}$derived in the previous sub-section must satisfy.

By construction, the extremal transaction $\hat{a}$ has the properties

 0 is an IRR of $\hat{a}$

 $\acute{NPV(\hat{a}, 0)}$ $\geq $ 0

 $τ$($\hat{a}$) = $τ$(**a**) . (For convenience, the value $τ$(**a**) will simply be denoted as $τ$.)

It is now shown that these conditions place some quite stringent conditions on the individual terms of the transaction.

Let $\hat{a}$ be the transaction (-a, b,………, -c, d), where a, b, c, d > 0. Then the above conditions mean that the following equations must hold

 – a + b – c + d =0 (4)

 b – (n-1)c + nd = - $ε$ for some $ε \geq 0 .$ (5)

 bc = $τ$ad (6)

Now, equation (4) implies that

 c = - a + b + d (7)

substituting into (5) implies

 b - (n-1)(-a + b + d) + nd = -$ ε$

 i.e., d = -(n-1)a + (n-2)b –$ ε$ (8)

substituting (8) into (7) implies

 c = - a + b -(n-1)a + (n-2)b –$ ε$

 i.e. c = -na + (n-1)b –$ ε$ (9)

Substituting from (8) and (9) into (6) implies that

 b[ -na + (n-1)b –$ ε]=$ $τ$a[-(n-1)a + (n-2)b –$ ε]$

 i.e. $τ$(n-1)$a^{2}$ + (n-1)$b^{2}$ –[n+$ τ$(n-2)]ab + $ε$($τ$a – b) = 0 (10)

Now, going back to equation (5), that equation implies

 - (n-1)c + nd = -b - $ε$ < 0 ,

hence nd < (n-1)c , and so d < c ;

hence, from (6), $τ$a = $\frac{bc}{d}$ > b: i.e., ($τ$a – b) > 0.

So equation (10) implies that

 $τ$(n-1)$a^{2}$ + (n-1)$b^{2}$ –[n+$ τ$(n-2)]ab $\leq $ 0 .

Dividing through by $a^{2}$ that implies

 (n-1)$(\frac{b}{a})^{2}$ - [n+$ τ$(n-2)]$(\frac{b}{a})$ + $τ$(n-1) $\leq $ 0 (11)

This implies that $(\frac{b}{a})$ must lie between the two roots of the quadratic equation

 (n-1)$x^{2}$ - [n+$ τ$(n-2)]$x$ + $τ$(n-1) = 0, (12)

provided that real roots for this equation exist.

The first thing to do, therefore, is to investigate under what conditions real roots for equation (12) do exist. Doing this will in fact enable us to complete the proof of Theorem 1. The requirement for real roots in (12) is that

 $[n+ τ(n-2)]^{2}$ - 4$ τ(n-1)^{2}$ $\geq $ 0 :

Applying the formula for the difference of two squares to the left hand side, the sign of the left hand side is the same as the sign of

 $n+ τ(n-2)$ - 2$ τ^{0.5}(n-1)$ .

So, writing y = $ τ^{0.5}$ , what is required is that

 (n-2)$y^{2}$ – 2(n-1)y + n $\geq $ 0.

It is easy to verify that the two roots of the quadratic on the left are y=1, and y = $\frac{n}{(n-2)}$ .The quadratic in y is non-negative if y is less than or equal to the smaller root, or greater than or equal to the larger. In terms of $τ$, therefore, this requires $τ \leq 1$, or $τ \geq $ $\frac{n^{2}}{(n-2)^{2}}$ . But we know that $τ$ > 1, since $τ $is the $τ$ function of our original transaction, **a**.

What has been established is that the quadratic at (12) has real roots only if $τ \geq $ $\frac{n^{2}}{(n-2)^{2}}$ .

That is, a necessary condition for $\hat{a}$to have an IRR at 0 for which the slope of the net present value function is non-negative is that $τ \geq $ $\frac{n^{2}}{(n-2)^{2}}$ . But, as has been proved, if the starting transaction **a** has multiple IRRs this implies that the extremal transaction $\hat{a}$must satisfy these conditions. Turning this round, this implies that if $τ$(**a**) < $\frac{n^{2}}{(n-2)^{2}}$ , then **a** has a unique IRR.

To complete the proof of Theorem 1, all that remains is to exclude the possibility that a transaction might exist with multiple IRRs, and $τ=$ $\frac{n^{2}}{(n-2)^{2}}$ . But if $τ=$ $\frac{n^{2}}{(n-2)^{2}}$ the expression on the left of (11) is always positive, except when $(\frac{b}{a})$ is equal to $\frac{n}{(n-2)}$ , when the expression on the left of (11) is equal to zero. Since strict equality in (11) only occurs when the original transaction **a** is itself extremal, the only possible transaction with multiple IRRs and $τ=$ $\frac{n^{2}}{(n-2)^{2}}$ is extremal, and is in fact the transaction (-1, $\frac{n}{(n-2)}$ , ……, $\frac{-n}{(n-2)}$ , 1), or a constant multiple of this. It is not difficult to show that this transaction has a triple IRR at 0, and so, by Descartes rule of signs, can have no other IRR. Therefore, the sufficient condition for **a** to have a unique IRR can be tightened to $τ$(**a**) $\leq $ $\frac{n^{2}}{(n-2)^{2}}$ , so completing the proof of Theorem 1.

Which leaves Theorem 2, the proof of which is completed in the next sub-sections.

6) Now assume that $τ>\frac{n^{2}}{(n-2)^{2}}$ : then, as established above, the quantity $(\frac{b}{a})$ must lie between the two roots of the quadratic at (12). These roots are

 $ \frac{\left[n+ τ\left(n-2\right)\right] \pm [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}{2(n-1)}$ .

Now the quantity $(\frac{b}{a})$, by construction, is equal to (1 + $π\_{1}$), where, (as will be recalled from the beginning of sub-section 3), $π\_{1}$ is the IRR of the first term in the unique partition of **a**. But the **a** we are dealing with was obtained by scaling an original transaction with multiple IRRs by a scaling factor $ρ$ where $ρ$ was an IRR of the original transaction satisfying the property that the slope of the NPV function at $ρ$ was non-negative. By the previous theory on scaling, therefore,

 $(\frac{b}{a})$ = $\frac{(1+ σ\_{1})}{(1+ ρ)}$ , where we now use $σ\_{1}$ to denote the IRR of the first term in the unique partition of the original transaction.

So what has been established is that, in terms of the original transaction,

$\frac{\left[n+ τ\left(n-2\right)\right] - [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}{2(n-1)}$ $\leq $ $\frac{(1+ σ\_{1})}{(1+ ρ)}$ $\leq $ $\frac{\left[n+ τ\left(n-2\right)\right] + [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}{2(n-1)}$ ,

or, equivalently,

$\frac{2(n-1)(1+ σ\_{1})}{(n+ τ\left(n-2)\right)+ [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}} \leq \left(1+ ρ\right)\leq $ $\frac{2(n-1)(1+ σ\_{1})}{(n+ τ\left(n-2)\right) - [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}$

where $σ\_{1}$ is the IRR of the first term in the unique partition of the original transaction.

Since any transaction with multiple IRRs must have at least one IRR where the slope of the NPV function at that IRR is non-negative, this establishes part (iii) of Theorem 2. It also follows that every transaction has at most one IRR which is strictly greater than the bound implied by part (i) of Theorem 2: or strictly smaller than the bound implied by part (ii), since no transaction can have two consecutive IRRs at each of which the slope of the NPV function is negative.

7) As well as proving part (iii), what has also been proved so far, therefore, is a partial proof of parts (i) and (ii) of Theorem 2, in the sense that parts (i) and (ii) hold if the inequality signs in those parts are replaced by strict inequality signs.

In order to complete the proof of Theorem 2 suppose that there exists a transaction **a** for which equality holds in either (i) or (ii) of the theorem, and the NPV of **a** has $\acute{NPV(a, ρ)}$ $\geq $ 0 . Then if the process is gone through of scaling by $ρ$ , and then constructing the extremal transaction $\hat{a}$ as in the proof above, it follows that $\hat{a}$ must have the property that the quantity $(\frac{b}{a})$ must exactly equal one or other of the roots of the quadratic at (12) above. Further, equality must hold at (11): but the only way equality can hold at (11), as has been proved above, is if the transaction **a** itself is extremal.

If we denote by r either root of the quadratic at (12), and if we use equations (8) and (9), it follows that $\hat{a}$ must, up to a constant multiple, be equal to the transaction

 (-1, r, ………, [n – (n-1)r], [-(n-1)+ (n-2)r]) .

It is easy to check that 0, as expected, is an IRR of $\hat{a}$ .

Further, $\acute{NPV(\hat{a}, 0)}$ = -r – (n-1)[n – (n-1)r] - n[-(n-1) + (n-2)r] = 0. So 0 is in fact a repeated IRR of $\hat{a}$ .

Also, the second derivative of NPV( $\hat{a}, u)$ at 0 is equal to

 (n-1)(n-2)[n – (n-1)r] + n(n-1)[-(n-1) + (n-2)r] .

Therefore the sign of the second derivative is the same as the sign of

 (n-2)[n – (n-1)r] + n[-(n-1) + (n-2)r]

 = -n + (n-2)r .

So the second derivative of the NPV function at 0 is > or < 0 according as r is > or < $\frac{n}{(n-2)}$ .

Now, if we denote by f(x) the quadratic at (12), then

 f($\frac{n}{(n-2)}$) = (n-1)$(\frac{n}{n-2})^{2 }-\left[n+τ\left(n-2\right)\right](\frac{n}{n-2})+τ(n-1)$

 = (-n + n -1)$τ$ + $\frac{n^{2}}{(n-2)}[\frac{\left(n-1\right)}{\left(n-2\right)}-1]$

 = - $τ$ + $\frac{n^{2}}{(n-2)^{2}}$

 < 0 .

Hence, since the quadratic is negative at $\frac{n}{(n-2)}$ , it follows that $\frac{n}{(n-2)}$ must lie between the two roots of the quadratic.

Let us now consider first of all the case when r = $\frac{\left[n+ τ\left(n-2\right)\right] + [(n+ τ\left(n-2)\right)^{2}- 4 τ\left(n-1\right)^{2}]^{0.5}}{2(n-1)}$ : that is, when r is the larger root of the quadratic, (which corresponds to the Theorem 2 (ii) case). It follows that r > $\frac{n}{(n-2)}$ , and so the second derivative of the NPV function of $\hat{a}$ is positive.

This implies that NPV($\hat{a}$, x) > 0 for some x > 0. But since the NPV function is negative for large u, this implies $\hat{a}$ must have an IRR at some value > 0. Since $\hat{a}$ has at most three IRRs, and the IRR at 0 is repeated, that implies $\hat{a}$ can have no IRR which is less than 0. Hence, since we know **a** is itself an extremal transaction, neither can **a** have an IRR less than $ρ$. In other words, the possibility has been ruled out that there could be any transaction with one IRR less than the implicit bound in 2(ii), and another IRR equal to it. So the strict inequality which has been already proved for 2(ii) can be strengthened to less than or equals, as in the statement of the theorem.

A corresponding argument can be used in case (i): in this case, since we are dealing with the smaller root of the quadratic, it follows that the second derivative is negative at 0. This concludes the proofs of Theorems 1 and 2.

**Appendix 3: Proof of the limiting values in d) of Section 4.**

It is immediately obvious that the lower bound tends to zero.

The limiting value of the upper bound depends on the limit, as $τ $ tends to infinity, of the lower root of the quadratic at (12) in sub-section 5 of the proof of Theorems 1 and 2 . This limit is $\frac{(n-1)}{(n-2)}$ . This can be proved directly: or, more easily, can be seen directly from equation (12) itself. If, as in sub-section 7 of the proofs of Theorems 1 and 2, the quadratic at (12) is denoted as f(x), then it can readily be seen that

 f( $\frac{(n-1)}{(n-2)})= \frac{(n-1)}{(n-2)^{2}}$ > 0 for all$ τ$ :

but, for $ε>0, $ f( $\frac{(n-1)}{(n-2)}+ ε)= \frac{(n-1)}{(n-2)^{2}}$ + $2ε\frac{(n-1)^{2}}{(n-2)}$ + $ε^{2}\left(n-1\right)$ -$ εn $- $ ε(n-2)τ$ :

this latter expression is negative for large enough $τ$ . Hence the smaller root of the quadratic can be made arbitrarily close to $\frac{(n-1)}{(n-2)}$ for large $τ$, hence showing that this is the limiting value.

The limiting value of the upper bound in Theorem 2 follows immediately.

**Appendix 4: Proof of Theorem 3 in Section 5.**

The proof deals with the two possible cases.

Case 1: At the largest IRR, $ρ$ , of **a**, the NPV function has a local maximum.

Note that this includes the case where **a** has a unique IRR.

For $ε>0$, define the new transaction $\hat{a}$($ε$) , of length n+2, as follows:-

 $\hat{a\_{j}}$ = $ a\_{j}$ , j = 0, ….,n.

 $\hat{a\_{n+1}}$ = $ε(1+ρ)^{n+1}$ .

Then $ \hat{a}$ satisfies the conditions of Theorems 1 and 2, so Theorem 2 can be applied to $\hat{a}$ .

Now NPV($\hat{a}$, $ρ$) =NPV(**a**, $ρ$) + $ε$ = $ε$ > 0.

But since, by reducing $ε$, NPV($\hat{a}$,u) can be made arbitrarily close to NPV(**a**, u) in a neighbourhood of $ρ$ , it follows that, for small enough $ε$, NPV($\hat{a}$, x) must be < 0 for some x < $ρ$, and in the neighbourhood of $ρ$.

Hence $\hat{a}$($ε$) must have at least one IRR between x and $ρ$ : let $ρ(ε)$ be the largest such: clearly, as $ε$ $\rightarrow $ 0, $ρ(ε)$ $\rightarrow $ $ρ$.

But since $\hat{a}$($ε$) must have an IRR which is greater than $ρ$, part (iii) of Theorem 2 implies that

 1 + $ρ(ε)$ $\leq $ $\frac{2n(1+ σ\_{1})}{(n+1+ τ\left(n-1)\right) - [(n+1+ τ\left(n-1)\right)^{2}- 4 τ\left(n\right)^{2}]^{0.5}}$ , where the $τ$ in this expression = $τ(\hat{a}$($ε$)).

As $ε$ tends to zero, the left hand side of this inequality tends to 1 + $ρ$: also as $ε$ tends to zero, the IRR of the last transaction in the unique partition of $\hat{a}$($ε$) clearly tends to -1, so $τ(\hat{a}$($ε$)) tends to infinity. Applying the result at (d) in Section 4, (and remembering that $\hat{a}$($ε$) is a transaction of length n+2), the right hand side tends to $(1- \frac{1}{n})(1+ σ\_{1})$ . Since $ρ$ is the largest IRR of **a**, in this case what has been proved is that all IRRs of **a** must be less than or equal to the bound implicit in the theorem.

Case 2: At the largest IRR, $ρ\_{MAX}$ , of **a**, the NPV function does not have a local maximum.

In this case, let $ρ$ be the second largest IRR of **a**. Note that NPV(**a**, x) > 0 for $ρ$ < x < $ρ\_{MAX}$ .

Define the transaction $\hat{a}$($ε$) , of length n+2, as follows:-

 $\hat{a\_{j}}$ = $ a\_{j}$ , for j < n.

 $\hat{a\_{n}}$ = $a\_{n}$ - 2$ε(1+ρ)^{n}$ .

 $\hat{a\_{n+1}}$ = $ε(1+ρ)^{n+1}$ .

Then NPV($\hat{a}$, $ρ$) =NPV(**a**, $ρ$) -2$ ε$ + $ε$ = -$ε$ < 0.

But since, by reducing $ε$, NPV($\hat{a}$, x) can be made arbitrarily close to NPV(**a**, x) on the interval [$ρ$, $ρ\_{MAX}$], for small enough $ε$ there must be some x in this interval such that NPV($\hat{a}$, x) is positive. Hence $\hat{a}$($ε$) must have at least one IRR between $ρ$ and x: let $ρ(ε)$ be the smallest such: then, as $ε$ $\rightarrow $ 0, $ρ(ε)$ $\rightarrow $ $ρ$.

Since $\hat{a}$($ε$) must have an IRR which is greater than x, part (i) of Theorem 2 applies, and the proof of this case of Theorem 3 goes through as in Case 1 above.

Note

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