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CATEGORISATION OF ADDITIVE PURCHASING POWER PARITIES

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Additivity is an important property for the aggregation methods used in constructing purchasing power parities. For a practical definition of additivity, this paper categorises all aditive methods. First of all, a generalisation of the Geary-Khamis method of aggregation is defined: this is called the Generalised Geary-Khamis, (GGK), approach. The key result proved is that, within a broad class of possible aggregation methods, the set of additive methods is precisely equivalent to the set of GGK indices. Some implications of this categorisation of additive methods are considered, both in the multilateral and bilateral cases. For example, in the multilateral case, the Iklé index is set in context as another special case of the GGK approach. In the bilateral case, it is shown that there always exists a GGK, (and therefore additive), equivalent to the Fisher Index.

1. INTRODUCTION

(1.1) Consider a system in which there are J countries, and I items, (or commodities). Let p_{ij} and q_{ij} denote, respectively, the price and quantity of commodity i in the jth country, where prices are expressed in the national currency of each country.

(1.2) Given data on prices and quantities, the aggregation problem is concerned with how to define purchasing power parities, and the real volumes of country Gross Domestic Products (GDP).

More formally, the purchasing power parity of country j, (PPP_j), is defined as follows. Suppose that, for every pair of countries j and k, the aggregation method produces an estimate, v_{jk}, of the ratio of the real volume of GDP between the two countries, where the quantities v_{jk} are multiplicative, in the sense that $v_{jk}v_{kl} = v_{jl}$ for all countries j, k and l. Then, for any pair of countries j and k, the ratio of the purchasing power parities between j and k is defined to be the expenditure ratio divided by the volume ratio: that is

$$\frac{\mathrm{PPP}_j}{\mathrm{PPP}_k} = \sum_i p_{ij}q_{ij} \Big/ \sum_i p_{ik}q_{ik} \cdot \frac{1}{v_{jk}}.$$

This determines the quantities PPP_j up to a multiplicative constant. If one country is (arbitrarily) selected as the numeraire, or base, country, and if the PPPs are scaled so that the PPP of the numeraire country is set to 1, then the PPPs of all countries are determined in absolute terms. The purchasing power parity of the jth country (PPP_j) then represents the number of jth country currency units that are equivalent in purchasing power to one unit of the numeraire currency.

In fact, for convenience, it is usually easier to work with the reciprocal of PPP_j. This is denoted by e_j, and represents the number of numeraire or base currency units that are equivalent in purchasing power to one unit of the jth currency. For an introduction to some standard methods of aggregation, and a useful, if now somewhat dated, review of issues, see Hill (1982).

(1.3) Many aggregation methods define real volumes in terms of international prices. Let π_i denote the international price of commodity i: then the real volume of gross domestic product in country j is defined to be $\sum_i \pi_i q_{ij}$.

The use of real volumes defined in terms of international prices offers great convenience, since it enables sub-aggregate volumes to be readily computed for any sub-group of commodities: and it also enables such sub-aggregates to be added up, or compared, between countries. This paper is concerned primarily with those aggregation methods which compute international prices. Not all methods of aggregation, however, lead to the computation of international prices. For example, the Gini–Elteto–Koves–Schultz (GEKS) method, (formerly commonly referred to as the Elteto–Koves–Schultz, or EKS method), proceeds directly to the computation of volume relativities between countries at aggregate GDP level.

(1.4) The structure of this paper is as follows. Section 2 defines the general class of aggregation methods to be considered. Section 3 introduces the concept of additivity. Section 4 defines a generalisation of the standard Geary-Khamis (GK) method of aggregation, denoted here as the generalised Geary-Khamis (GGK) approach. Section 5 proves the fundamental results on the categorisation of additive aggregation methods—showing that GGK methods are the only additive methods. Sections 6 and 7 consider some implications for the general multi-country case, and the bilateral comparison case respectively. In particular, in Section 6, a specific GGK is illustrated, which gives equal weight, in defining international prices, to each of the countries in the comparison: and in Section 7, it is proved that, for a bilateral comparison, it is always possible to find a specific GGK which gives a volume relativity between the two countries exactly equal to the standard Fisher volume index. Finally, Section 8 contains illustrations of the application of specific GGK indices to simple data.

2. The Class of Aggregation Methods Considered

(2.1) This paper is concerned with the class of aggregation methods which satisfy the following two properties:

1. International prices are a weighted arithmetic average of individual country prices, where the individual country prices are first of all converted to a comon currency unit by price deflators e_j.
2. The price deflators e_j are proportional to the ratios between country real volumes, and country expenditures.

(2.2) These properties determine a set of simultaneous equations which the quantities π_i and e_j must satisfy. More formally, the class of aggregation methods considered in this paper is defined as follows, (where **P** and **Q** are arbitrary matrices of positive prices and quantities):

Let the weight matrix $\mathbf{A} = \mathbf{A}[\mathbf{Q}] = [\alpha_{ij}]$ be a matrix of positive elements, where the individual elements may be functions of **Q**: assume **A** is normalised so that its first row sums to 1.

International prices, and country price deflators are defined in terms of the following equations in π_i, e_j and λ.

(1) $$\pi_i = \sum_j e_j \alpha_{ij} p_{ij} \quad \text{for all } i$$

(2) $$e_j = \lambda\left(\sum_i \pi_i q_{ij} \Big/ \sum_i p_{ij} q_{ij}\right) \quad \text{for all } j$$

provided that positive solutions π_i, e_j and λ always exist.

(2.3) Different specifications for the weight matrix $\mathbf{A}$ correspond to different aggregation methods within this broad class. For example,

(1) if $\alpha_{ij} = g_j$ for all i, this corresponds to a class of indices introduced by Van Ijzeren: see, for example, Van Ijzeren (1987). In particular, when $\alpha_{ij} = 1/J$, then $\mathbf{A}$ corresponds to a simple unweighted Van Ijzeren index.

(2) If α_{ij} is defined as $\alpha_{ij} = q_{ij}/(\sum_j q_{ij})$, then it is a standard result that solutions always exist to equations (1) and (2) with $\lambda \equiv 1$, and this particular $\mathbf{A}$ corresponds to the Geary-Khamis aggregation method [see Geary (1958)].

(2.4) Note that, in the above defining equations, the normalisation constraint $\sum_j \alpha_{ij} = 1$ does not affect the generality of the definition: it is necessary to impose some normalisation constraint on $\mathbf{A}$, or λ is not determined up to a multiplicative constant.

(It might seem more natural to impose the stronger constraint on $\mathbf{A}$, that each of its rows sums to 1. From the point of view of the theory developed in this paper, however, this would be an unnecessarily restrictive assumption, since the fundamental categorisation result, Theorem 2 below, requires only that the weaker assumption be made.)

(2.5) Note also that the π_i and e_j are not determined up to a multiplicative constant: in other words, if π_i and e_j are solutions to the above equations, and if π_i and e_j are all multiplied by an arbitrary positive constant, then the transformed values are still solutions to the equations. As noted in 1.2 above, it is often convenient to use this in order to normalise π_i and e_j so that $e_j = 1$ for the base country. On occasion in some of the proofs in this paper it will be more convenient to adopt a different normalisation: this will be indicated wherever this happens.

3. Additivity, and the Additivity Conjecture

(3.1) One important property which an aggregation method may possess is known as additivity. In common usage, the term "additivity" is sometimes applied in a fairly loose sense, to mean any aggregation method for which absolute volumes can be calculated and added up, over subsets of items, or over countries. In this sense, any aggregation method satisfying equation (2) above might be loosely described as additive. In this paper, however, a more precise concept of additivity is employed. Firstly, attention is restricted in this paper to aggregation methods where the calculation of international prices is by means of a weighted arithmetic mean formula, rather than, for example, a weighted geometric or harmonic mean. Secondly, following the approach in Rao (1997), it is required for additivity as considered here that aggregate real domestic product, derived by converting national aggregate expenditure using PPPs, should be equal to the aggregate derived through valuation of country quantities at international prices. In algebraic terms, this is equivalent, as Rao notes, to the requirement that

$$(3) \qquad e_j \sum_i p_{ij} q_{ij} = \sum_i \pi_i q_{ij} \quad \text{for all } j,$$

which is equivalent to equation (2) above, with $\lambda = 1$.

What is required is that this condition holds for all possible positive price and quantity matrices **P** and **Q**. Formally, therefore, the definition of additivity used in this paper, which might be denoted "strong additivity," is as follows:

Let **A** be an aggregation method defined as in 2.2 above, and suppose that positive solutions π_i and e_j always exist to equations (1) and (2), with $\lambda \equiv 1$: then **A** is defined to be additive.

Throughout the rest of this paper, additivity refers to strong additivity defined in this way.

(3.2) Note that, even for systems which are not additive, volume relativities calculated directly between countries are the same as those calculated as the ratio of deflated expenditures. This can be readily seen, since

$$\frac{\sum_i \pi_i q_{ij}}{\sum_i \pi_i q_{ik}} = \frac{\lambda e_j \sum_i p_{ij} q_{ij}}{\lambda e_k \sum_i p_{ik} q_{ik}} = \frac{e_j \sum_i p_{ij} q_{ij}}{e_k \sum_i p_{ik} q_{ik}}.$$

However, for many purposes, it is more convenient to work with absolute volumes, rather than volume relativities: and when an aggregation method is not additive in the sense defined in 3.1 above, inconsistencies will arise between aggregate volumes calculated directly, and those calculated from deflated expenditures. The importance of additivity for an aggregation method, therefore, is that additivity guarantees that such inconsistencies cannot arise.

(3.3) In the defining equations of the GK system, $\lambda = 1$, by definition. The GK system is therefore additive in the sense defined in 3.1 above. In the original paper on the GK index, (Geary, 1958), Geary noted that the defining equations of the GK system inmplied that the following identity held: namely,

$$\sum_i \sum_j e_j p_{ij} q_{ij} \equiv \sum_i \sum_j \pi_i q_{ij}$$

(This expression is Geary's identity 3, with minor changes in notation.) Geary stated that "The essential property of the system is, of course, the identity 3 which seems to be the analogue of the circular test (and a circular test which is fulfilled) in index number theory."

The above identity follows immediately from the definition of additivity given in 3.1 above. Accordingly, it follows that systems which are additive in the sense used in this paper satisfy Geary's "essential property."

(3.4) In his interesting 1997 paper reviewing various aspects of the theory of aggregation, Rao (1997) posed a conjecture about additivity: namely that, apart from the Geary-Khamis index itself, it would be "nearly impossible" to define other indices which were additive. This paper is concerned with answering this conjecture.

4. Generalisation of the Geary-Khamis System

(4.1) As already noted, if the weight matrix **A** is defined by $\alpha_{ij} = q_{ij}/(\sum_j q_{ij})$, then this defines the standard GK system.

Let **b** be a vector of positive quantities β_j. Then a natural generalisation of the GK is to define **A** by

$$\alpha_{ij} = \beta_j q_{ij} \Big/ \sum_j \beta_j q_{ij} \tag{4}$$

In what follows, any aggregation method defined as in equation (4) is denoted a Generalised Geary-Khamis (GGK) method.

(4.2) One application in the literature which, it turns out, is an example of a specific GGK index, is the index proposed by Iklé in her 1972 paper. The Iklé index corresponds to a GGK in which the quantities β_j are taken to be inversely proportional to country volumes: this will be discussed further in Section 6 below. Another example of the suggested use of an index which is actually a particular member of the GGK class was pointed out to me by Professor Khamis, in commenting on an early version of this paper: namely that, at an expert group on PPPs convened by Eurostat and the International Association for Research on Income and Wealth in 1982, he had suggested the use of what amounts to a special case of the GGK index, with β_j inversely proportional to country populations.

5. Categorisation of Additive PPPs

(5.1) Given the above preliminaries, then the basic results on the categorisation of additive PPPs are contained in the following two theorems.

Theorem 1. If **A** is a GGK aggregation method, then **A** is additive.

Proof. Let **A** be defined by the vector **b** and let **P** and **Q** be any price and quantity matrices. Then to establish the result, it is sufficient to prove that the following equations always have positive solutions π_i and e_j: namely

$$\pi_i = \sum_j e_j \alpha_{ij} p_{ij} \quad \text{for all } i \tag{1'}$$

$$e_j = \sum_i \pi_i q_{ij} \Big/ \sum_i p_{ij} q_{ij} \quad \text{for all } j \tag{2'}$$

Define a new quantity matrix **Q′** by $q'_{ij} = \beta_j q_{ij}$.

Then the above equations can be written equivalently, in terms of **P** and **Q′** as

$$\pi_i = \sum_j e_j q'_{ij} p_{ij} \Big/ \sum_j q'_{ij} \quad \text{for all } i$$

$$e_j = \sum_i \pi_i q'_{ij} \Big/ \sum_i p_{ij} q'_{ij} \quad \text{for all } j$$

These are the defining equations for the GK method applied to **P** and **Q′**: hence, by the results established by Khamis (1972), on the existence of solutions to the Geary-Khamis equations, positive solutions exist. This establishes the desired result.

The converse result, that GGK indices are the only additive ones, is proved in the following theorem. Let **A** satisfy the conditions set out in 2.2 above: then

Theorem 2. If **A** defines an additive aggregation method, then **A** is GGK.

Proof. See annex for proof of Theorem 2.

(5.2) These results show that the set of additive methods, as defined in 3.1 above, is equivalent to the set of GGK methods: and also answer the conjecture on additivity noted in 3.4 above.

(5.3) In the above discussion, attention has been restricted to GGK indices where each of the individual components β_j is strictly positive: and to weight matrices **A** where the individual terms α_{ij} are also strictly positive. In fact, it is not difficult to prove that corresponding results still hold when some, (but not all), of the terms β_j, and the corresponding columns of **A**, are zero.

6. Implications: The Equal Scaled Country Volume Approach

(6.1) The above categorisation of additive PPPs has potentially important implications for multilateral comparisons—not all of which have yet been thought through. What it means is that there is a wide class of additive methods to choose from—the class of GGK methods. In any particular application, it may therefore be possible to choose a specific GGK index which has suitable properties for that specific problem.

(6.2) For example, the standard GK method is sometimes criticised because countries with large volumes will have large weight in the calculation of international prices. Hence, if there is a dominant country in the comparison, GK international prices will tend to follow the pattern of prices in that country—which, it is sometimes argued, may lead to distortions in resulting volume comparisons—[see, for example, Hill (1982), for discussion on this point.]

(6.3) If this is regarded as a problem, then the GGK approach opens up the possibility of defining an additive aggregation method which gives equal weight to each of the countries in the comparison. In the GGK, the volume q_{ij} is replaced for weighting purposes by the scaled volume $\beta_j q_{ij}$: hence a natural definition of a GGK which gave equal weight to each country would involve finding a vector **b** such that

$$\sum_i \pi_i \beta_j q_{ij} = K \quad \text{for all } j. \tag{5}$$

It is convenient to denote any GGK satisfying equation (5) as an "equal scaled country volume" aggregation.

As noted in 4.2 above, the index proposed by Iklé (1972), is a GGK, with weights β_j inversely proportional to a measure of country volumes. The Iklé index is, in fact, identical to the equal scaled country volume index described above:

this is not, however, immediately obvious, given the rather difficult notation employed by Iklé. This paper will, therefore, proceed to derive some of the properties of the equal scaled country volume index from first principles and will establish later that this index is exactly equivalent to Iklé's.

(6.4) The equal scaled country volume approach, therefore involves finding solutions for the following equations in π_i, e_j and β_j: namely

$$\pi_i = \sum_j e_j \beta_j q_{ij} p_{ij} \Big/ \sum_j \beta_j q_{ij} \quad \text{for all } i \tag{6}$$

$$e_j = \sum_i \pi_i q_{ij} \Big/ \sum_i p_{ij} q_{ij} \quad \text{for all } j \tag{7}$$

$$\sum_i \pi_i \beta_j q_{ij} = K \quad \text{for all } j. \tag{8}$$

(In these equations, the constant K is arbitrary, since, in the GGK, the parameters β_j are indeterminate up to a multiplicative constant.)

From equation (8) it follows that

$$\beta_j = K\left[\sum_i \pi_i q_{ij}\right]^{-1} \tag{9}$$

in other words, in the equal scaled country volume case, the scaling factors β_j are inversely proportional to the country volumes—as would be expected.

From equations (7) and(9)

$$e_j \beta_j = K \Big/ \sum_i p_{ij} q_{ij} \quad \text{for all } j. \tag{10}$$

From equation (6)

$$\sum_j \pi_i \beta_j q_{ij} = \sum_j e_j \beta_j q_{ij} p_{ij} \quad \text{for all } i \tag{11}$$

Substituting (9) into the left side of (11), and (10) into the right side, implies that the equal scaled country volume π_is must satisfy

$$\sum_j (\pi_i q_{ij}) \Big/ \sum_k \pi_k q_{kj} = \sum_j (p_{ij} q_{ij}) \Big/ \sum_k p_{kj} q_{kj} \quad \text{for all } i \tag{12}$$

Equation (12) is important for a number of reasons.

First, it is possible, (but tedious), to show that a positive solution to (12) always exists.

Secondly, equation (12) can be used to justify the assertion, made in the preceding paragraph, that the Iklé index is identical to the equal scaled country volume index. (This follows, since a simple re-ordering of (12) can be readily seen to be equivalent to the defining equations for Iklé's international prices, on p. 203 of her paper.)

Thirdly, equation (12) also implies that the equal scaled country volume international prices satisfy an intuitively reasonable condition: namely, when the share of a given item in each country is calculated at international prices, and aggregated over countries, (that is, the left side of the equation), that is the same as the share of the item in each country at individual country prices, aggregated over countries, (i.e. the right side of the equation).

The equal scaled volume approach will not be considered further in this paper, but it is one illustration of how it may be possible to choose specific GGK indices to have properties which might be desirable for particular problems.

(6.5) Another example of choosing a specific GGK to have particular properties is discussed in the next section, which looks at the two country case, where it is shown that there always exists a GGK which gives a volume relativity exactly equal to the Fisher volume index.

(6.6) An open question, which would merit further research, is to consider how closely the GEKS volume relativities between countries could be approximated by an appropriate choice of GGK index: and under what circumstances a GGK could be found which was exactly equal to the GEKS.

7. The GGK Approach in the Bilateral Case

(7.1) In the bilateral case, (i.e. when $J=2$), it is relatively easy to derive an explicit expression for the GGK index, as follows.

It is a standard result that, in the bilateral a case, the ordinary GK index has

$$\frac{e_1}{e_2}=\sum_i \frac{p_{i2}q_{i1}q_{i2}}{q_{i1}+q_{i2}} \Big/ \sum_i \frac{p_{i1}q_{i1}q_{i2}}{q_{i1}+q_{i2}}.$$

Applying this formula to $\mathbf{P}$ and $\mathbf{Q}'$ as defined in the proof of Theorem 1 above, implies that, for the GGK in the bilateral case,

$$(13)\quad \frac{e_1}{e_2}=\sum_i \frac{p_{i2}\beta_1 q_{i1}\beta_2 q_{i2}}{\beta_1 q_{i1}+\beta_2 q_{i2}} \Big/ \sum_i \frac{p_{i1}\beta_1 q_{i1}\beta_2 q_{i2}}{\beta_1 q_{i1}+\beta_2 q_{i2}}=\sum_i \frac{p_{i2}q_{i1}q_{i2}}{\beta_1 q_{i1}+\beta_2 q_{i2}} \Big/ \sum_i \frac{p_{i1}q_{i1}q_{i2}}{\beta_1 q_{i1}+\beta_2 q_{i2}}.$$

Explicit expressions for the other quantities of interest in the bilateral case can readily be derived from (13).

(7.2) In the case of bilateral comparisons, the Fisher volume index, (that is, the geometric average of the Paasche and Laspeyres volume indices), is often regarded as having ideal properties. One of the weaknesses of the Fisher index, however, is that it is not defined in terms of international prices. In fact, the following argument shows that there is always a GGK index which exactly equals the Fisher: thus, there is a set of additive international prices which exactly recreates the Fisher. The proof depends on a continuity argument.

Let P^V_{12} and P^P_{12} be the Paasche volume and price indices between countries 1 and 2, defined respectively as

$$P^V_{12}=\sum_i p_{i1}q_{i1} \Big/ \sum_i p_{i1}q_{i2} \quad \text{and} \quad P^P_{12}=\sum_i p_{i1}q_{i1} \Big/ \sum_i p_{i2}q_{i1}.$$

Let L_{12}^V and L_{12}^P be the Laspeyres volume and price indices between countries 1 and 2, defined respectively as

$$L_{12}^V = \sum_i p_{i2}q_{i1} \Big/ \sum_i p_{i2}q_{i2} \quad \text{and} \quad L_{12}^P = \sum_i p_{i1}q_{i2} \Big/ \sum_i p_{i2}q_{i2}.$$

The Fisher volume and price indices are defined as

$$F_{12}^V = [P_{12}^V L_{12}^V]^{1/2} \quad \text{and} \quad F_{12}^P = [P_{12}^P L_{12}^P]^{1/2};$$

and it follows readily that

$$F_{12}^V F_{12}^P = \sum_i p_{i1}q_{i1} \Big/ \sum_i p_{i2}q_{i2}\text{: i.e. that}$$

$$F_{12}^V = \left(\sum_i p_{i1}q_{i1} \Big/ \sum_i p_{i2}q_{i2}\right) F_{21}^P.$$

Now, for a GGK index,

$$\sum_i \pi_i q_{i1} / \sum_i \pi_i q_{i2} = \left(\sum_i p_{i1}q_{i1} / \sum_i p_{i2}q_{i2}\right)\frac{e_1}{e_2}.$$

This implies that, to find a GGK index which recreates the Fisher volume index, it is sufficient to find a **b** such that, for the GGK index corresponding to this **b**,

$$\frac{e_1}{e_2} = F_{21}^P.$$

For β in the range [0, 1], define

$$G_{12}(\beta) = \sum_i \frac{p_{i2}q_{i1}q_{i2}}{\beta q_{i1} + (1-\beta)q_{i2}} \Big/ \sum_i \frac{p_{i1}q_{i1}q_{i2}}{\beta q_{i1} + (1-\beta)q_{i2}}.$$

Equation (13) implies that $G_{12}(\beta) = e_1/e_2$, for the GGK with $\mathbf{b} = [{}_{(1-\beta)}^{\beta}]$. But $G_{12}(\beta)$ is a continuous function of β on [0, 1], with $G_{12}(0) = L_{21}^P$, and $G_{12}(1) = P_{21}^P$: so, by continuity, there exists at least one β such that $G_{12}(\beta) = F_{21}^P$, hence establishing the desired result.

Note that the GGK index which recreates the Fisher is not necessarily unique among GGK indices. Note also that there is at least one other, non-additive, set of international prices which also recreates the Fisher, as illustrated, for example, in Van Ijzeren (1987).

8. Illustrative Calculations

(8.1) This final section contains some illustrations of calculated GGK indices, using simple data: (4 items and 3 countries). The data is entirely artificial, and the calculations are intended primarily to illustrate that the algebra of the GGK approach actually works as the theory implies.

TABLE 1

Part A: Illustrative Data						
	Prices			Quantities		
	"G.B."	"U.S."	"FR"	"G.B."	"U.S."	"FR"
Rice	5	12	20	50	95	40
Eggs	2	6	13	200	300	210
Fish	3	5	11	300	700	400
Steel	10	15	42	90	250	50

Part B: Results			
Real Volumes	INDEX 1	INDEX 2	INDEX 3
"G.B."	4,562	4,527	4,544
"U.S."	10,190	10,190	10,190
"FR"	4,365	4,308	4,337
International Prices			
Rice	10.56	9.90	10.24
Eggs	5.26	5.00	5.12
Fish	5.06	5.05	5.06
Steel	16.26	16.86	16.56
Price Deflators			
"G.B."	1.862	1.848	1.855
"U.S."	1.000	1.000	1.000
"FR"	0.435	0.430	0.432
Scaling Factors, β_j			
"G.B."	1	2.251	1.5
"U.S."	1	1	1
"FR"	1	2.365	1.5

Part C: Check on Additivity for Index 3			
Real Volumes			
	"G.B."	"U.S."	"FR"
Rice	511.8	972.5	409.5
Eggs	1,023.7	1,535.5	1,074.9
Fish	1,518.4	3,543.0	2,024.6
Steel	1,490.0	4,139.0	827.8
Total	4,544.0	10,190.0	4,336.7
Deflated expenditures			
	"G.B."	"U.S."	"FR"
Rice	463.7	1,140.0	345.9
Eggs	741.9	1,800.0	1,180.4
Fish	1,669.2	3,500.0	1,902.4
Steel	1,669.2	3,750.0	908.0
Total	4,554.0	10,190.0	4,336.7

The data is set out in Table 1, Part A, and has been constructed so that one country, (the "U.S."), is dominant in volume terms in each of the four commodities: and also so that the price of the commodity in which the U.S. is relatively largest, (steel), is relatively smallest in the U.S. To this extent, therefore, this data illustrates the kind of inverse relationship between relative price and quantity which might be expected in the real world.

(8.2) Three indices have been calculated using this data, as follows:

Index 1: the standard Geary-Khamis index.

Index 2: the equal scaled country volume, or Iklé, index.

Index 3: the particular GGK index defined by $\beta_1 = 1.5$, $\beta_2 = 1$, $\beta_3 = 1.5$.

Table 1, Part B sets the results for these three indices applied to the data in Part A, showing the resulting country volumes, international prices and price deflators. Also shown are the relevant β_j factors for each index. The U.S. has been taken as the base country.

(8.3) It can readily be verified that these values do satisfy the additivity conditions. For example, Table 1, Part C demonstrates for Index 3 that real volumes and deflated expenditures sum to the same total for each country.

(8.4) Note that the real volumes for both G.B. and Fr in Table 1, Part B are higher for Index 1, (the GK), than for Index 2, (the equal scaled country volume index). This is as expected, given the particular structure built into the illustrative data set, (with steel having a relatively low price and high volume in the U.S.). It can be seen that, for this data, the equal scaled country volume index implies weighting quantities in both G.B. and Fr by something over a factor of 2 (see the relevant β_js in Table 1, Part B). It turns out, therefore, that for this data, the particular GGK chosen as Index 3 represents an intermediate position roughly halfway between the GK and the equal scaled country volume index (this is, of course, only a chance result for this particular set of data—not a general result).

9. Conclusion

(9.1) This paper has categorised those aggregation methods which are additive in the sense defined in 3.1 above, showing that additive aggregation methods are precisely equivalent to the class of indices defined here as Generalised Geary-Khamis indices. One important consequence of this categorisation is that it opens up a broad field of potential candidates, (the GGK indices), within which suitable additive solutions for particular problems may be sought. Two examples have been given to illustrate this approach, the equal scaled country volume, or Iklé, index, and the GGK equivalent to the Fisher. A particularly fruitful field for further research is likely to be to identify further examples of GGK indices with specific properties. For example, one question posed, but not solved, is to consider under what circumstances there exists a GGK equivalent to the GEKS index.

Annex: Proof of Theorem 2

Let **P** and **Q** be arbitrary matrices of positive prices and quantities. Then since **A** is additive, the following property holds: namely, the equations

(A1) $$\pi_i = \sum_j e_j \alpha_{ij} p_{ij} \quad \text{for all } i$$

(A2) $$e_j = \sum_i \pi_i q_{ij} \Big/ \sum_i p_{ij} q_{ij} \quad \text{for all } j$$

always have positive solutions π_i and e_j.

In what follows the implications of the above property are pursued, with **Q** held fixed, and for various choices of **P**. The proof proceeds by establishing a number of Lemmas.

Lemma 1. If $p_{ij}/(p_{kj}) \leq \varepsilon$ for all j, then there exists $K>0$ such that $\pi_i/(\pi_k) \leq K\varepsilon$, where K does not depend on $\mathbf{P}$.

Proof.

$$\frac{\pi_i}{\pi_k} = \sum_j e_j \alpha_{ij} p_{ij} \Big/ \sum_j e_j \alpha_{kj} p_{kj} = \sum_j \left(\frac{\alpha_{ij} p_{ij}}{\alpha_{kj} p_{kj}}\right) e_j \alpha_{kj} p_{kj} \Big/ \sum_j e_j \alpha_{kj} p_{kj}$$

This is a weighted average of the terms in round brackets, so

$$\frac{\pi_i}{\pi_k} \leq \varepsilon . \max_j \left(\frac{\alpha_{ij}}{\alpha_{kj}}\right),$$

establishing the Lemma.

Lemma 2. Let $\mathbf{D}$ be a $[2 \times 2]$ matrix: and suppose that, for any $\varepsilon > 0$, there exists a vector $\mathbf{p} = [\begin{smallmatrix}\pi_1\\ \pi_2\end{smallmatrix}]$, with $\pi_1 = 1$, and a vector $\mathbf{y} = [\begin{smallmatrix}y_1\\ y_2\end{smallmatrix}]$, with $\max(|y_1|, |y_2|) \leq \varepsilon$, such that $\mathbf{Dp} = \mathbf{y}$. Then $\mathbf{D}$ is singular.

Proof. Suppose $\mathbf{D}$ is non-singular: then, since $\mathbf{Dp} = \mathbf{y}$,

$$\mathbf{p} = \mathbf{D}^{-1}\mathbf{y} = \frac{1}{|\mathbf{D}|}\begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix}\mathbf{y}.$$

Therefore, $1 = \pi_1 = 1/|\mathbf{D}|\,(y_1 d_{22} - y_2 d_{12})$. Therefore, $||\mathbf{D}|| \leq |y_1||d_{22}| + |y_2||d_{12}| \leq 2\varepsilon \max(|d_{22}|, |d_{12}|)$. Since ε is arbitrary, this implies $|\mathbf{D}| = \mathbf{0}$: hence contradiction, and so $\mathbf{D}$ is singular.

Lemma 3. Let $\mathbf{P}$ be any matrix of positive prices, (of dimension $I \times J$). Define $E_j = p_{1j}q_{1j} + p_{2j}q_{2j}$, for all j. Then the following identity holds

$$\left(1 - \sum_j \frac{\alpha_{1j} p_{1j} q_{1j}}{E_j}\right)\left(1 - \sum_j \frac{\alpha_{2j} p_{2j} q_{2j}}{E_j}\right) - \left(\sum_j \frac{\alpha_{1j} p_{1j} q_{2j}}{E_j}\right)\left(\sum_j \frac{\alpha_{2j} p_{2j} q_{1j}}{E_j}\right) = 0.$$

Proof. Take $\varepsilon > 0$: define a modified matrix $\mathbf{P}$ by replacing every element of $\mathbf{P}$ from the third row down by ε: i.e. $p_{ij} = \varepsilon$ for all $i \geq 3$. Let π_i and e_j be the solutions of equations (A1) and (A2) for the modified $\mathbf{P}$. Since π_i and e_j are indeterminate up to a multiplicative constant, it is convenient, for present purposes to assume that π_i and e_j have been scaled so that $\pi_1 = 1$.

Let $E'_j = \sum_i p_{ij} q_{ij}$. Then $E'_j = E_j + O(\varepsilon)$, (where this notation denotes that there exists some constant C, not dependent on ε, such that $|E'_j - E_j| \leq C\varepsilon$). Also, by Lemma 1, $\pi_i \leq K\varepsilon$ for all $i \geq 3$.

Therefore,

$$\pi_1 = \sum_j e_j \alpha_{1j} p_{1j} = \sum_j \alpha_{1j} p_{1j} \sum_i \frac{\pi_i q_{ij}}{E'_j}$$

$$= \pi_1 \sum_j \frac{\alpha_{1j} p_{1j} q_{1j}}{E'_j} + \pi_2 \sum_j \frac{\alpha_{1j} p_{1j} q_{2j}}{E'_j} + O(\varepsilon)$$

$$= \pi_1 \sum_j \frac{\alpha_{1j} p_{1j} q_{1j}}{E_j} + \pi_2 \sum_j \frac{\alpha_{1j} p_{1j} q_{2j}}{E_j} + O(\varepsilon):$$

that is,

$$\left(1-\sum_j \frac{\alpha_{1j}p_{1j}q_{1j}}{E_j}\right)\pi_1 - \left(\sum_j \frac{\alpha_{1j}p_{1j}q_{2j}}{E_j}\right)\pi_2 = O(\varepsilon).$$

A similar argument, starting with the definition of π_2, shows that

$$-\left(\sum_j \frac{\alpha_{2j}p_{2j}q_{1j}}{E_j}\right)\pi_1 + \left(1-\sum_j \frac{\alpha_{2j}p_{2j}q_{2j}}{E_j}\right)\pi_2 = O(\varepsilon).$$

Since ε is arbitrary, Lemma 2 applies, (where the matrix **D** is defined by the coefficients of π_1 and π_2 in the last two equations). Hence **D** is singular, giving the required identity.

Lemma 4. **A** is stochastic: i.e. $\sum_j \alpha_{ij} = 1$ for all i.

Proof. Given the initial normalisation of **A**, $\sum_j \alpha_{1j} = 1$. To prove **A** is stochastic, it is required to show that $\sum_j \alpha_{kj} = 1$ for all $k \neq 1$. Since the ordering of items is arbitrary, it is enough to prove this for $k = 2$.

In the identity in Lemma 3, take $p_{1j} = \varepsilon$, $p_{2j} = [q_{2j}]^{-1}$, for all j. Then, as $\varepsilon \to 0$, the left side of the identity tends to $(1 - \sum_j \alpha_{2j})$, establishing the lemma.

Lemma 5. The following identity holds

$$\left(\sum_j \frac{\alpha_{1j}q_{2j}}{q_{1j}}\right)\left(\sum_j \frac{\alpha_{2j}q_{1j}}{q_{2j}}\right) = 1.$$

Proof. In the basic identity in Lemma 3, define

$$p_{1j} = [q_{1j}]^{-1}, p_{2j} = [q_{2j}]^{-1}, \quad \text{for all } j.$$

Then

$$\frac{p_{1j}q_{1j}}{E_j} = \frac{p_{2j}q_{2j}}{E_j} = \frac{1}{2}, \quad \text{for all } j.$$

So the identity in Lemma 3 then implies

$$\left(1-\frac{1}{2}\sum_j \alpha_{1j}\right)\left(1-\frac{1}{2}\sum_j \alpha_{2j}\right) - \frac{1}{4}\left(\sum_j \frac{\alpha_{1j}q_{2j}}{q_{1j}}\right)\left(\sum_j \frac{\alpha_{2j}q_{1j}}{q_{2j}}\right) = 0$$

since **A** is stochastic, (Lemma 4), this implies

$$\frac{1}{4} - \frac{1}{4}\left(\sum_j \frac{\alpha_{1j}q_{2j}}{q_{1j}}\right)\left(\sum_j \frac{\alpha_{2j}q_{1j}}{q_{1j}}\right) = 0,$$

giving the required result.

Lemma 6. For all i, j, k, the following identity holds

$$\text{(A3)} \qquad \frac{\alpha_{ij}}{\alpha_{kj}} = u_{ik} \cdot \frac{q_{ij}}{q_{kj}}$$

where

$$u_{ik} = \sum_n \frac{\alpha_{in} q_{kn}}{q_{in}}.$$

Proof. Without loss of generality, this is proved for $i = 1$, $k = 2$, $j = 1$. Take λ in the range $0 < \lambda < 1$. In the basic identity in Lemma 3, take

$$\begin{aligned} p_{11} &= \lambda (q_{11})^{-1} \\ p_{21} &= (1-\lambda)(q_{21})^{-1} \\ p_{1j} &= (q_{1j})^{-1} \quad \text{for all } j \geq 2 \\ p_{2j} &= (q_{2j})^{-1} \quad \text{for all } j \geq 2. \end{aligned}$$

Then the identity implies that

$$\left[1 - \lambda\alpha_{11} - \frac{1}{2}\sum_{j>1} \alpha_{1j}\right]\left[1 - (1-\lambda)\alpha_{21} - \frac{1}{2}\sum_{j>1} \alpha_{2j}\right]$$
$$- \left[\frac{\lambda\alpha_{11} q_{21}}{q_{11}} + \frac{1}{2}\sum_{j>1} \frac{\alpha_{1j} q_{2j}}{q_{1j}}\right]\left[\frac{(1-\lambda)\alpha_{21} q_{11}}{q_{21}} + \frac{1}{2}\sum_{j>1} \frac{\alpha_{2j} q_{1j}}{q_{2j}}\right] = 0.$$

That is,

$$\left[1 - \lambda\alpha_{11} - \frac{1}{2}(1-\alpha_{11})\right]\left[1 - (1-\lambda)\alpha_{21} - \frac{1}{2}(1-\alpha_{21})\right]$$
$$- \left[\frac{\lambda\alpha_{11} q_{21}}{q_{11}} + \frac{1}{2}\sum_{j>1} \frac{\alpha_{1j} q_{2j}}{q_{1j}}\right]\left[\frac{(1-\lambda)\alpha_{21} q_{11}}{q_{21}} + \frac{1}{2}\sum_{j>1} \frac{\alpha_{2j} q_{1j}}{q_{2j}}\right] = 0.$$

Let u denote $\sum_j (\alpha_{1j} q_{2j}/q_{1j})$, (and note that, by Lemma 5, $u^{-1} = \sum_j (\alpha_{2j} q_{1j}/q_{2j})$.) Then the above equation can be written

$$\frac{1}{2}[1 + \alpha_{11}(1-2\lambda)]\frac{1}{2}[(1 - \alpha_{21}(1-2\lambda)]$$
$$- \frac{1}{2}\left[-\frac{(1-2\lambda)\alpha_{11} q_{21}}{q_{11}} + u\right]\frac{1}{2}\left[\frac{(1-2\lambda)\alpha_{21} q_{11}}{q_{21}} + u^{-1}\right] = 0,$$

or equivalently

$$1 + \alpha_{11}(1-2\lambda) - \alpha_{21}(1-2\lambda) - \alpha_{11}\alpha_{21}(1-2\lambda)^2$$
$$+ \alpha_{11}\alpha_{21}(1-2\lambda)^2 - \frac{u\alpha_{21} q_{11}(1-2\lambda)}{q_{21}} + \frac{u^{-1}\alpha_{11} q_{21}(1-2\lambda)}{q_{11}} - 1 = 0$$

i.e.

$$\left(\alpha_{11} - \alpha_{21} - \frac{u\alpha_{21} q_{11}}{q_{21}} + \frac{u^{-1}\alpha_{11} q_{21}}{q_{11}}\right)(1-2\lambda) = 0.$$

Since this holds for arbitrary λ, it follows that the expression in the first bracket in this equation must be identically equal to 0. In other words,

$$\alpha_{11}\left(1+\frac{u^{-1}q_{21}}{q_{11}}\right)-\alpha_{21}\left(1+\frac{uq_{11}}{q_{21}}\right)=0;$$

i.e.

$$\frac{\alpha_{11}}{\alpha_{21}}=\left(1+\frac{uq_{11}}{q_{21}}\right)\bigg/\left(1+\frac{u^{-1}q_{21}}{q_{11}}\right)=\frac{uq_{11}}{q_{21}},$$

hence establishing the lemma.

Given the above lemmas, the proof of the theorem can now be concluded as follows. From equation (A3) above, it follows readily that the quantities u_{ik} defined in Lemma 6 satisfy $u_{ik}u_{kn}=u_{in}$, for all i, k and n. Hence there exist quantities

$$\gamma_i>0 \quad \text{such that} \quad u_{ik}=\frac{\gamma_i}{\gamma_k}.$$

Hence, from Lemma 6, and for any j, it follows that

$$\frac{\alpha_{1j}}{\gamma_1 q_{1j}}=\frac{\alpha_{2j}}{\gamma_2 q_{2j}}=\frac{\alpha_{3j}}{\gamma_3 q_{3j}}\cdots=\frac{\alpha_{Ij}}{\gamma_I q_{Ij}}=\beta_j, \text{say}$$

i.e. $\alpha_{ij}=\gamma_i\beta_j q_{ij}$, for all i and j. But since $\sum_j \alpha_{ij}=1$, this implies that $\gamma_i=(\sum_j \beta_j q_{ij})^{-1}$, and hence $\alpha_{ij}=\beta_j q_{ij}/\sum_j \beta_j q_{ij}$, thus completing the proof of the theorem.

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